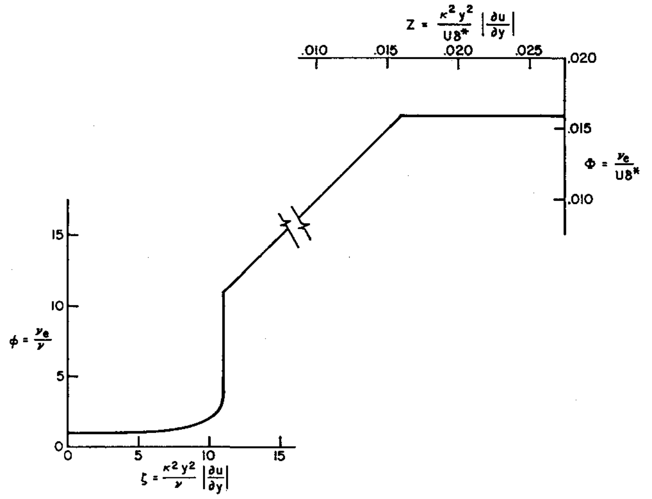

Fig. 1 The complete effective viscosity function. Note that $\varphi(\zeta) = R\Phi(Z)$, $\zeta = RZ$ where $R = U\delta^*/\nu$. The overlap portion increases with R.

small wall shear stress τ_0 and incipient separation. Although not stated explicitly, the analysis of Refs. 1 and 4 is limited to values of $\nu(dp/dx)/\rho\, u_\tau^3$ less than about 0.05; physically this corresponds to the disappearance of the logarithmic portion of the law of wall.

The basis of the work reported in Refs. 2 and 3 is a completely defined hypothesis for the effective or eddy viscosity ν_e. Thus,

$$\nu_e/\nu = \varphi[(\kappa^2 y^2/\nu)(\partial u/\partial y)] \qquad \text{for small } y \qquad (1a)$$

$$\nu_e/U\delta^* = \Phi[(\kappa^2 y^2/U\delta^*)(\partial u/\partial y)] \qquad \text{for large } y \qquad (1b)$$

If an overlap region exists where both functions apply, then it follows that $\nu_e = \nu\varphi = U\delta^*\Phi = \kappa^2 y^2\, \partial u/dy$. This, of course, is a result identical to that given by Prandtl's mixing length theory. Furthermore, $\varphi \rightarrow 1$ as $y \rightarrow 0$, and the complete function φ may be obtained with data at zero pressure gradient. For large enough y, $\Phi = 0.016$ in accordance with Clauser's hypothesis for the outer layer. Figure 1 summarizes the complete hypothesis, the basis of which is considered in more detail in Ref. 3.

The complete equations of motion have been solved numerically. A slight, a posteriori, approximation† is made so that the final results appear directly in the following simple forms: for the defect layer,

$$(U - u)/u_\tau = f'[(y/\Delta), \beta] \qquad (2a)$$

and for the wall layer,

$$u/u_\tau = u^+ [(u_\tau y/\nu), \alpha] \qquad (2b)$$

where

$$\Delta \equiv \int_0^\infty \left[\frac{(U - u)}{u_\tau}\right] dy, \qquad \beta \equiv \frac{\delta^*}{\tau_0}\frac{dp}{dx}$$

in accordance with Clauser's usage; $u_\tau \equiv (\tau_0/\rho)^{1/2}$ and $\alpha = \nu(dp/dx)/\rho\, u_\tau^3$.

It should be noted that the functions f' and u^+ automatically overlap for a significant portion of the boundary layer and not at a single point. The overlap portion is not necessarily logarithmic.

In the overlap region, we find that $U/u_\tau = (2/c_f)^{1/2} = f' + u^+ = fcn(\beta, U\delta^*/\nu)$ so that a comparatively simple skin-friction equation is directly obtained.

All of the preceding equations may be presented in a form that remains finite as $\beta \rightarrow \infty$ or $u_\tau \rightarrow 0$.

Although analytically more precise, and without benefit of empirical adjustment, as β (or the parameter S of Ref. 4) is varied, agreement with data is better than that obtained in Ref. 4. Furthermore, it was possible to predict the data of Stratford for $1/\beta = 1/\alpha = 0$.

Equilibrium turbulent boundary layers are described simply and directly by the condition $\beta(x) \equiv \delta^*\,(dp/dx)/\tau_0 =$ const and without recourse to streamwise integrations of the momentum integral equation. Furthermore, it is determined that equilibrium turbulent boundary layers exist in the range $-0.5 \leq \beta \leq \infty$ (the lower limit actually varies slightly with $U\delta^*/\nu$).

It is believed that the work described in Refs. 2 and 3 is a precursor to a general turbulent boundary-layer theory that correctly identifies separation as the point where $\tau_0(x) = 0$. Further extensions to heat and mass transfer are readily foreseeable and will be pursued.

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† We actually find that $(U - u)/u_\tau = f'\,(y/\Delta, \beta, u_\tau/U)$, but the slight dependency on u_τ/U may be eliminated within a known and negligible error.

Nonequilibrium Sodium Ionization in Laminar Boundary Layers

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Introduction

An ablation heat shield in its manufacture can be contaminated with very small amounts of highly ionizable compounds, which, although insignificant with respect to their total weight, can become dominant in the production of electrons. These trace contaminants can then determine the observable characteristics of a re-entry vehicle. It is the purpose of this note to describe a method for determining the boundary-layer ionization due to trace contaminants.

Basic Equation and Solution

Consider a contaminant being injected into the boundary layer along a flat plate in small enough amounts so that it does not significantly alter the velocity and enthalpy profiles that will be determined by the gross amount of blowing. If the ions, atoms, and electrons associated with the trace contaminant are treated as one species, its distribution through the boundary layer can be represented by the following similar profile:

$$S = \int_0^\eta [f''(\eta)]^{Sc} d\eta \Big/ \int_0^\infty [f''(\eta)]^{Sc} d\eta \qquad (1)$$

where

$$S = (C_{Aw} - C_A)/C_{Aw} \qquad (2)$$

and where Sc is the Schmidt number. The governing species

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equation for the similar distribution of ions through the boundary layer is

$$2\frac{l}{Sc}\frac{d^2C_i}{d\eta^2} + f\frac{dC_i}{d\eta} + \frac{2x\dot{w}_i}{\rho u_e} = 0 \tag{3}$$

where

$$\dot{w}_i = \dot{w}_i(\eta)$$

and the factor 2 implies simultaneous diffusion of ions and electrons.[1]

Boundary Conditions:

$$\text{at} \quad \eta = 0 \quad C_i = C_{iw}$$
$$\eta = \infty \quad C_i = 0$$

The solution to Eq. (3) is

$$C_i = C_{iw} + A\int_0^\eta [f''(\eta)]^{Sc/2}\,d\eta - \frac{x}{u_e}\frac{Sc}{l}\int_0^\eta [f''(\eta)]^{Sc/2}\int_0^\eta \frac{\dot{w}_i}{\rho}[f''(\eta)]^{-Sc/2}d\eta d\eta \tag{4}$$

where

$$A = \frac{\dfrac{x}{u_e}\dfrac{Sc}{l}\displaystyle\int_0^\infty [f''(\eta)]^{Sc/2}\int_0^\eta \frac{\dot{w}_i}{\rho}[f''(\eta)]^{-Sc/2}d\eta d\eta - C_{iw}}{\displaystyle\int_0^\infty [f''(\eta)]^{Sc/2}\,d\eta} \tag{5}$$

Ionization Kinetics

Sodium was studied as a trace contaminant. The particular ionization reactions considered and their rate constants are[2]

	$k(T)$	
$Na + M \rightarrow Na^+ + e^- + M$	$80T^{3/2}\,e^{-(107{,}500/T)}$	(6)
$Na + e^- \rightarrow Na^+ + 2e^-$	$3.1(10)^{11}T^{1/2}e^{-(107{,}500/T)}$	(7)
$Na + 2O \rightarrow Na^+ + e^- + O_2$	$1.026(10)^{13}$	(8)

where M refers to any third body and T is in degrees Rankine.

Only the forward direction is considered for each reaction, implying that the reactions are not close to equilibrium. An equilibrium calculation can be made to check the validity of this assumption. The equilibrium between an atom and its ions can be described by

$$\epsilon = [K_p/(P + K_p)]^{1/2} \tag{9}$$

where the equilibrium constant K_p is described by[3]

$$\log K_p = -(5040\,E/T) + \tfrac{5}{2}\log T - 6.5 + \log(g_e g_A{}^+/g_A) \tag{10}$$

From the described ionization reactions and the law of mass action, the source term is

$$\frac{\dot{w}_i}{\rho} = \rho C_{Na}\left\{\frac{k_1}{M_m}(1 - C_O) + k_2\frac{C_e{}^-}{M_e{}^-} + k_3\,\rho\frac{(C_O)^2}{M_O{}^2}\right\} \tag{11}$$

where k_i is the rate constant associated with reaction i and M_i is the molecular weight of species i.

To a first approximation, the distribution of sodium atoms C_{Na} is given by Eq. (1). Bortner's values[2] were used for the rate constants. The atomic oxygen mass fraction C_O is a known function of η.[4]

The source term is completely described as a function of η except for the temperature dependency of the rate constants.

For a Lewis number of unity and a constant Prandtl number, the energy equation is (assuming similarity)

$$fg' + \frac{l}{Pr}g'' + \frac{u_e{}^2 l}{h_{se} - h_{sw}}\left(1 - \frac{1}{Pr}\right)\frac{d}{d\eta}(f'f'') = 0 \tag{12}$$

where

$$g = \frac{h_s(\eta) - h_{sw}}{h_{se} - h_{sw}} \tag{13}$$

For a constant wall temperature, the solution to Eq. (12) is

$$g(\eta) = C_1\int_0^\eta [f''(\eta)]^{Pr}\,d\eta - \frac{Pr}{l}\int_0^\eta [f''(\eta)]^{Pr}\int_0^\eta H(\eta)[f''(\eta)]^{-Pr}\,d\eta d\eta \tag{14}$$

where

$$C_1 = \frac{Pr}{l}\int_0^\infty H(\eta)\,[f''(\eta)]^{-Pr}\,d\eta + \frac{1 - \dfrac{Pr}{l}\displaystyle\int_0^\infty H(\eta)[f''(\eta)]^{-Pr}\int_0^\eta [f''(\eta)]^{Pr}\,d\eta d\eta}{\displaystyle\int_0^\infty [f''(\eta)]^{Pr}\,d\eta} \tag{15}$$

$$H(\eta) = \frac{l\,u_e{}^2}{h_{se} - h_{sw}}\left(1 - \frac{1}{Pr}\right)\frac{d}{d\eta}\{f'f''\} \tag{16}$$

Equation (14) gives the temperature distribution as a function of η so that Eq. (4) can be used to describe the ion (and electron) distribution.

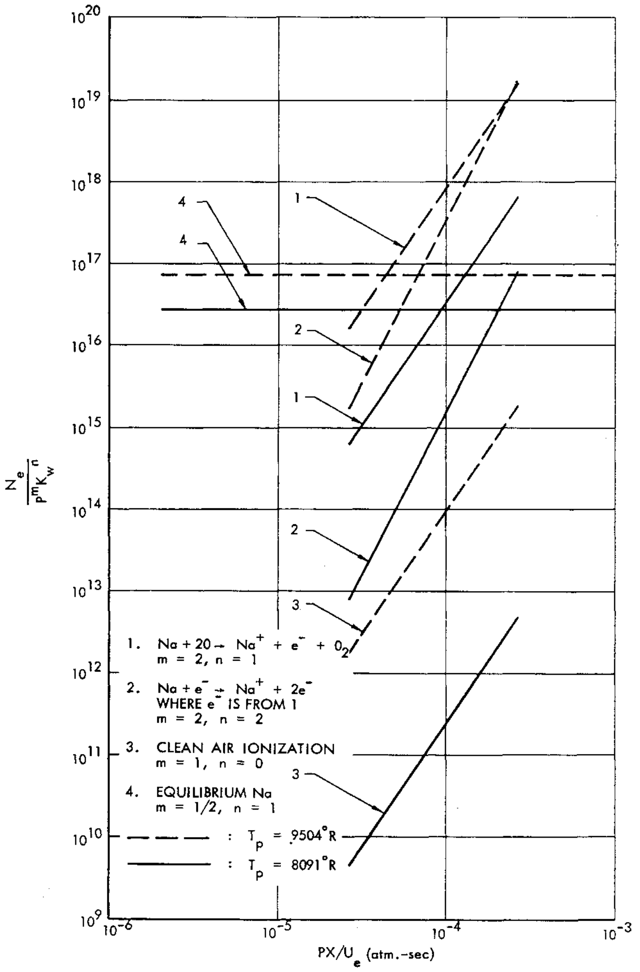

Fig. 1 **Nonequilibrium electron concentration.**

Results

Solutions to Eq. (4) were computed for each of the three ionizing reactions. The results are described in Fig. 1, which represents the scaling of electron density N_e vs. the parameter PX/u_e, where P is in atmospheres. K_w, the wall concentration of sodium, depends on the gross blowing rate as well as the level of heat-shield contamination. Included in Fig. 1 are the nonequilibrium clean air[4] and the equilibrium sodium ionization levels. To indicate the degree of temperature dependency, two cases with different peak temperatures were studied as indicated in Fig. 1.

The ionization from some of the reactions was very small so that all the results are not included in Fig. 1. The most significant reaction was found to be the ionization of sodium by collision with atomic oxygen. The other reactions produced negligible ionization in comparison.

If the clean air and sodium ionization is evaluated for a peak temperature of 8090°R, a wall contaminant concentration of 10^{-4}, a pressure of 0.186 atm., and a value of PX/u_e of $0.262\ (10)^{-4}$ atm-sec, the clean air, nonequilibrium sodium, and equilibrium sodium ionization levels, respectively, are found to be $8.8\ (10)^8$, $2.2\ (10)^9$, and $1.2\ (10)^{12}$ electrons/cm.³ In general, it was found that wall contaminant concentrations of 10^{-4} produced ionization levels comparable to those of clean air. To insure negligible sodium ionization, a wall concentration of 10^{-6} is required.

If diffusion is neglected, the nonequilibrium sodium ionization can be shown to be described by

$$\frac{Ne}{P^2} = 7.1\ (10)^{69} \left(\frac{PX}{u}\right)^3 \frac{C_{\text{Na}}}{T} \exp\left\{-\frac{212600}{T}\right\} \tag{17}$$

where T is in degrees Rankine. Figure 2 compares this stream-tube analysis with the results based on the solution of the diffusion equation. Included also is a similar comparison for the clean air computation. It is seen that, in both cases, the stream-tube method overpredicts by roughly two orders of magnitude.

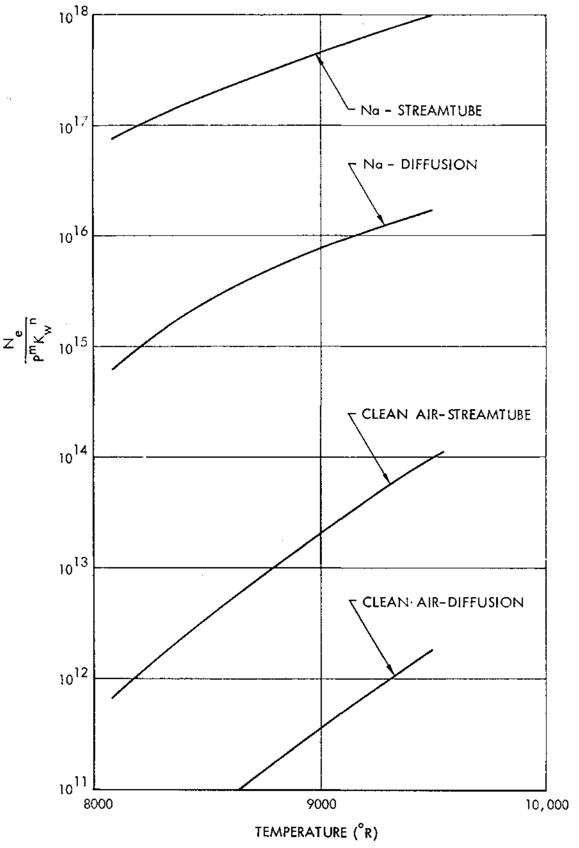

Fig. 2 Comparison of stream-tube and diffusion methods.

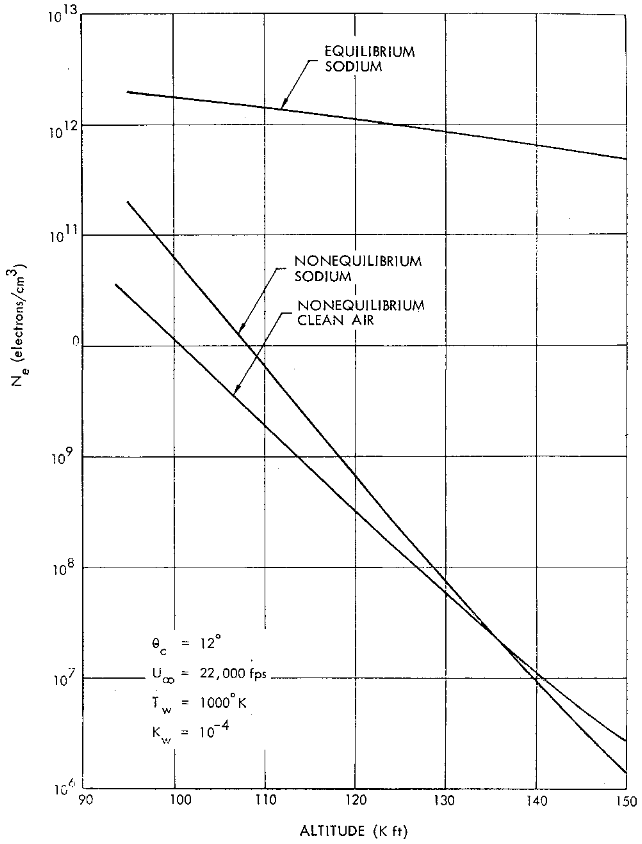

Fig. 3 Ionization variation with altitude.

Figure 3 describes the variations in clean air and sodium ionization with altitude for a 12° half-angle cone traveling at 22,000 fps with a wall temperature of 1000°K. The wall contaminant mass fraction was taken to be $(10)^{-4}$.

Conclusions

The following conclusions are based only on a study of the ionization reactions mentioned. Should other reactions be shown of significance, the same type of analysis could be performed.

1) Sodium nonequilibrium ionization can be determined by considering only the ionization of sodium by atomic oxygen.

2) For altitudes down to the vicinity of 100,000–90,000 ft, only the forward reactions need be considered. Below these altitudes, the sodium ionization will be close to equilibrium.

3) Sodium ionization will be significant with respect to clean air ionization depending primarily on the wall mass fraction of sodium atoms. A mass fraction of 10^{-6} will insure negligible sodium ionization.

4) Diffusion reduces the peak ionization in the boundary layer by approximately two orders of magnitude, whether one considers clean air or contaminant ionization.

5) Since the most significant sodium ionization reaction is a three-body collision, contaminant ionization is relatively more predominant at low altitudes.

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